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Phase Transitions in the Ising Model with Transverse Field

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We show the existence of a phase transition in the Ising model with transverse field for dimensions $\nu \geqslant 2$ provided the transverse term is sufficiently small. This is done by proving long-range order occurs using the reflection positivity of the Hamiltonian and localization of eigenvectors.

KEY WORDS: Ising model; phase transition; reflection positivity.

INTRODUCTION

In studying the problem of the existence of a phase transition for a statistical mechanical system one usually attempts to apply one of two general techniques. The first (historically) is a geometrical argument pioneered by Peierls[(1)] and made rigorous by Griffiths.[(2)] This method involves estimating the probability of certain configurations using contours which are naturally associated with those configurations. This method was developed for classical systems and was extended by Ginibre[(3)] to certain quantum lattice systems. He did this by adding a dimension (corresponding to time) to the lattice and applied the Trotter product formula to deal with the problem of noncommuting operators. The other approach, developed by Frohlich, Simon, and Spencer,[(4)] uses Fourier analysis to establish an infrared bound on certain thermodynamic functions. However, to employ this technique the Hamiltonian must be reflection positive.

The model that will be considered here is an Ising model perturbed by a transverse field. To describe this model, let Λ be a finite rectangular subset of

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Z^ν, where $\nu \geqslant 2$. For $i \in Z^\nu$ let $H(i)$ be C^2 regarded as a Hilbert space with the usual inner product. Define

$$H_\Lambda = \bigotimes_{i \in \Lambda} H(i) \tag{1.1}$$

Let $\sigma^x(i)$ and $\sigma^z(i)$ be the Pauli spin matrices acting on the site i:

$$\sigma^x(i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^z(i) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2}$$

For σ a configuration in $\{-1, 1\}^\Lambda$, set

$$|\sigma\rangle = \bigotimes_{i \in \Lambda} |\sigma(i)\rangle \tag{1.3}$$

where

$$|\sigma(i)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in H(i) \tag{1.4}$$

according to whether the spin at site i is up or down.

For the model under consideration, the Hamiltonian is given by

$$H_\Lambda = -\left[\sum_{\substack{i,j \in \Lambda \\ |i-j|=1}} \sigma^z(i)\,\sigma^z(j) + \varepsilon \sum_{i \in \Lambda} \sigma^x(i) \right] \tag{1.5}$$

For $A \in B(H_\Lambda)$, the thermodynamic expectation at inverse temperature β, $\langle \cdot \rangle_{\Lambda,\beta}$, is given by

$$\langle A \rangle_{\Lambda,\beta} = \frac{1}{\operatorname{Tr} e^{-\beta H_\Lambda}} \operatorname{Tr} A e^{-\beta H_\Lambda} \tag{1.6}$$

We will apply the concept of reflection positivity developed by Frohlich, Simon, and Spencer to show the existence of a phase transition in this model for ε small. In particular, we use the idea of exponential localization of eigenvectors developed by Frohlich and Lieb[5] to show the existence of long-range order for $\beta > \beta_c$ and $\varepsilon < \varepsilon_c$ with ε_c independent of β. It follows that the ground state for the model exhibits long-range order. A discussion of the final comment is contained in Ref. 6.

Remarks. This model has been considered by Ginibre,[3] and he has shown that spontaneous magnetization occurs in two dimensions for $\varepsilon \to 0$ as $\beta^{-1/2}$, $\beta \to \infty$. This result is counterintuitive in that at zero temperature ($\beta = \infty$), ε would have to be zero for spontaneous magnetization to occur. In

Ref. 7, the Peierls contour estimates which Ginibre obtained were used to analyze a slightly modified version of the transverse Ising model by operator theoretic techniques. This involved recognizing the changing length of the contours as time evolves as being a random walk on the positive integers. One then obtains a path space measure which may be estimated by using a Laplace transform. This technique gave only a slightly better results in that ε need go to zero only at the rate $\beta^{-1/3}$.

The reader is also referred to the work of Driessler, Landau, and Perez,[8] in which path methods are applied to the Ising model with transverse field to derive some correlation inequalities and obtain bounds on the spontaneous magnetization for dimensions $\nu \geqslant 3$.

2. REFLECTION POSITIVITY AND EXPONENTIAL LOCALIZATON OF EIGENVECTORS

In this section we follow methods developed by Frochlich and Lieb[5] to show that a phase transition occurs for ε small, independent of β in the transverse Ising model. We provide a sketch of the main ideas of their technique and show how these methods apply to the model under consideration. Recall the Hamiltonian

$$H_{\Lambda,\varepsilon} = -\left[\varepsilon \sum_{i\in\Lambda} \sigma^x(i) + \sum_{\substack{i,j\in\Lambda \\ |i-j|=1}} \sigma^z(i)\,\sigma^z(j)\right] \tag{2.1}$$

Theorem 2.1. Let $H_{\Lambda,\varepsilon}$ be as above and assume $\nu \geqslant 2$ where ν is the dimension of the lattice. Then there is an $M > 0$ and β_c such that $|\langle\sigma^z(0)\,\sigma^z(j)\rangle_\Lambda| \geqslant M$ for $\beta > \beta_c$ uniform in Λ, independent of j for ε sufficiently small. That is, long-range order occurs.

The proof given here is for the two-dimensional case. The extension to higher dimensions will be clear. First note that since

$$\sigma^z(i) = 1 - 2P^-(i) = 2P^+(i) - 1 \tag{2.2}$$

where $P^+(i)$ $[P^-(i)]$ is projection onto configuration with spin up (down) at site i, then

$$\langle\sigma^z(0)\,\sigma^z(j)\rangle_\Lambda = 1 - 4\langle P^-(0)\,P^+(j)\rangle_\Lambda \tag{2.3}$$

What will be shown is

$$\langle P^-(0)\,P^+(j)\rangle_\Lambda \leqslant \theta < \tfrac{1}{4} \tag{2.4}$$

independent of j, uniform in Λ.

Let p be a path made up of line segments joining nearest-neighbor sites such that p connects site 0 with site j. If σ is a configuration in which $\sigma(0) \neq \sigma(j)$ the path p must intersect at least one Peierls contour. Let $\hat{p}$ denote the midpoints of the lattice bonds which make up p and let $x \in \hat{p}$ be the point nearest to i where p intersects a Peierls contour that surrounds either site 0 or site j. Let $\Gamma(x, b, \alpha)$ denote a contour that surrounds site i or site j but not both, has shape α, length b, and intersects p at x. Let $P_{\Gamma(x,b,\alpha)}$ be projection onto configurations which have such a contour. Then

$$\langle P^-(0)\, P^+(j)\rangle_\Lambda \leqslant \sum_{x \in p} \sum_{b \leqslant 4} \sum_{\alpha} \langle P_{\Gamma(x,b,\alpha)}\rangle_\Lambda \tag{2.5}$$

The main step of the proof is to show that

$$\langle P_{\Gamma(x,b,\alpha)}\rangle_\Lambda \leqslant e^{-\mu b} \tag{2.6}$$

for μ sufficiently large, uniform in Λ. If this is the case, then the right-hand side (2.5) may be bounded as required by (2.4).

To prove inequality (2.6), note that for any contour Γ

$$\begin{aligned} \langle P_\Gamma\rangle_\Lambda &= \Big\langle \prod_{\langle i,j\rangle \in \Gamma} P^+(i)\, P^-(j) + \prod_{\langle i,j\rangle \in \Gamma} P^-(i)\, P^+(j)\Big\rangle_\Lambda \\ &= 2\Big\langle \prod_{\langle i,j\rangle \in \Gamma} P^+(i)\, P^-(j)\Big\rangle_\Lambda \end{aligned} \tag{2.7}$$

where $\langle i, j\rangle$ does a nearest-neighbor pair pair and $\langle i, j\rangle \in \Gamma$ iff the bond between i and j is intersected by Γ.

Lieb and Frohlich show that for *reflection positive* measures (which we define below)

$$\Big\langle \prod_{\langle i,j\rangle \in \Gamma} P^+(i)\, P^-(j)\Big\rangle_\Lambda \leqslant \langle p_\Lambda\rangle_\Lambda^{|\Gamma|/2|\Lambda|} \tag{2.8}$$

for some universal projector P_Λ. In our case P_Λ is projection onto the state which is perhaps best descirbed pictorially:

$$\begin{matrix} + & - & - & + & + & - & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & + & - & - & + \end{matrix}$$

where Λ is a cube with sides of length 2^M. We then adapt the Lieb–Frohlich *principle of exponential localization* to show that

$$\langle P_\Lambda \rangle_\Lambda^{|\Lambda|/2} \leqslant e^{-\mu} \tag{2.9}$$

where μ can be made arbitrarily large. Equations (2.7), (2.8), and (2.9) together imply the necessary inequality (2.6).

In the remaining part of this section we show that our Hamiltonian is reflection positive and how this leads to (2.9). Finally, we prove the key estimate (2.9) in Theorem 2.2 below.

We now review the concept of reflection positivity, which is a central idea in the proof. We restrict our attention to two dimensions and consider the sites to have half-integer coordinates. Divide the system about the line $x = 0$ to obtain $\Lambda_- = \{$sites whose first coordinate is negative$\}$ and $\Lambda_+ = \{$sites whose first coordinate is positive$\}$. If $\sigma \in \{-1, 1\}^\Lambda$ we can regard its restriction to $\{-1, 1\}^{\Lambda_-}$ by σ_-. Define $F_- = \{F \mid F(\sigma) = h(\sigma_-)$ for some $h\}$ and similarly for F_+. Let R: $\{-1, 1\}^{\Lambda_+} \to \{-1, 1\}^{\Lambda_-}$ by $R\sigma(i_x, i_y) = \sigma(-i_x, i_y)$ where $i = (i_x, i_y)$. We write $\sigma_- = R\sigma_+$. Define θ: $F_- \to F_+$ by

$$\theta F(\sigma) = \theta h(\sigma_+) = h(R\sigma_+) \tag{2.10}$$

Reflection positivity is the property that $\langle F\theta F \rangle_\Lambda \geqslant 0$ for $F \in F_-$.

We now show that the measure defined by

$$\mu_\epsilon(\sigma) = \exp[-\beta H_{\Lambda,\epsilon}(\sigma)] \tag{2.11}$$

is reflection positive. To see this, we write

$$H_{\Lambda,0} = - \sum_{\substack{i,j \in \Lambda \\ |i-j|=1}} \sigma^z(i)\, \sigma^z(j) \tag{2.12}$$

as

$$H_{\Lambda,0} = H_+ + \theta H_+ + \sum V_i \theta V_i \tag{2.13}$$

where H_+ involves terms to the right of $x = 0$ and $V_i \theta V_i$ are straddling terms. Now if $F \in F_+$ and $\langle\ \rangle_0$ denotes expection with respect to product measure then

$$\begin{aligned}
\langle \mu_0, F\theta F \rangle &= \frac{1}{Z_{\Lambda,0}} \left\langle F\theta F \exp\left(-\beta H_+ - \beta\theta H_+ - \beta \sum_i V_i \theta V_i\right) \right\rangle_0 \\
&= \frac{1}{Z_{\Lambda,0}} \sum_{n_i} \frac{\beta^{\Sigma_i n_i}}{\prod (n_i!)} \left\langle F e^{-\beta H_+} \prod_i V^{n_i} \theta \left(F e^{-\beta H_+} \prod_i V^{n_i} \right) \right\rangle_0 \\
&\geqslant 0
\end{aligned} \tag{2.14}$$

by expanding $\exp(V_i \theta V_i)$ in a power series. Since the transverse term K_Λ involves no straddling terms, $H_{\Lambda,\varepsilon}$ is reflection positive.

To prove inequality (2.8) let $\Gamma = \Gamma_V \cup \Gamma_H$ where $\Gamma_H = \{\langle i,j\rangle \in \Gamma |$ the contour between i and j is horizontal$\}$ and $\Gamma_V = \Gamma \backslash \Gamma_H$. Let $\Gamma_{H,e} = \{\langle i,j\rangle \in \Gamma_H |$ smaller y coordinate of i and j is even$\}$ and $\Gamma_{H,0} = \Gamma_H \backslash \Gamma_{H,e}$. Define similar sets $\Gamma_{V,0}$ and $\Gamma_{V,e}$. Thus

$$\Gamma = \Gamma_{H,0} \cup \Gamma_{H,e} \cup \Gamma_{V,0} \cup \Gamma_{V,e} \tag{2.15}$$

and the sets are pairwise disjoint. Applying the Schwarz inequality twice gives

$$\left\langle \prod_{\langle i,j\rangle \in \Gamma} P^+(i) P^-(j) \right\rangle_\Lambda \leqslant \prod_{\substack{\alpha = H,V \\ \beta = e,0}} \left\langle \prod_{\langle i,j\rangle \in \Gamma_{\alpha,\beta}} P^+(i) P^-(j) \right\rangle_\Lambda^{1/4} \tag{2.16}$$

Following Lieb and Frohlich, let $\mathcal{O}$ denote an arbitrary nonempty subset of $\Gamma_{H,e}$. Consider

$$\left\{ \left\langle \prod_{\langle i,j\rangle \in \mathcal{O}} P^+(i) P^-(j) \right\rangle^{1/2|\mathcal{O}|} : \mathcal{O} \subset \Gamma_{H,e} \right\}$$

Let

$$Z = \max_{\mathcal{O}} \left\{ \left\langle \prod_{\langle i,j\rangle \partial \mathcal{O}} P^+(i) P^-(j) \right\rangle^{1/2|\mathcal{O}|} \right\} \tag{2.17}$$

Using translation invariance and reflection positivity, by applying the Schwarz inequality repeatedly Lieb and Frohlich show

$$Z \leqslant \langle P_\Lambda \rangle_\Lambda^{1/|\Lambda|} \tag{2.18}$$

The following theorem provides the key estimate of $\langle P_\Lambda \rangle_\Lambda$.

Theorem 2.2. For any $\mu > 0$, there is a β_c and ε_c such that if $\beta > \beta_c$ and $\varepsilon < \varepsilon_c$ and $H_{\Lambda,\varepsilon}$ as given above, then

$$\langle P_\Lambda \rangle_{\Lambda,\varepsilon}^{1/|\Lambda|} < e^{-\mu} \tag{2.19}$$

Proof. Note that if $E_\Lambda(dk)$ is the spectral measure of $H_{\Lambda,\varepsilon}$ and $e_0(\varepsilon)$ is the infimum of the spectrum of $H_{\Lambda,\varepsilon}$, then

$$\begin{aligned} \langle P_\Lambda \rangle_{\Lambda,\beta,\varepsilon} = {} & \frac{1}{Z_{\Lambda,\varepsilon}(\beta)} \int_{e_0(\varepsilon)}^{e_0(\varepsilon) + \Delta|\Lambda|} e^{-\beta k} \operatorname{Tr}[E_\Lambda(dk)] P_\Lambda \\ & + \frac{1}{Z_{\Lambda,\varepsilon}(\beta)} \int_{e_0(\varepsilon) + \Delta|\Lambda|}^{\infty} e^{-\beta k} \operatorname{Tr}[E_\Lambda(dk)] P_\Lambda \end{aligned} \tag{2.20}$$

Here $\Delta > 0$ will be chosen later. The terms on the right-hand side of (2.20) will be denoted by $R_-(\beta, \Lambda)$ and $R_+(\beta, \Lambda)$, respectively.

We first treat $R_+(\beta, \Lambda)$. We have

$$R_+(\beta, \Lambda) \leqslant \frac{1}{Z_{\Lambda,\varepsilon}(\beta)} \exp\{-\beta[e_0(\varepsilon) + \Delta\,|\Lambda|]\} \int_{e_0(\varepsilon)+\Delta\,|\Lambda|}^{\infty} \operatorname{Tr}[E_\Lambda(dk)]$$

$$\leqslant \exp\{-\beta(e_0(\varepsilon) + \Delta\,|\Lambda|]\} \frac{\operatorname{Tr}(1)}{Z_{\Lambda,\varepsilon}(\beta)} \tag{2.21}$$

Now by Jensen's inequality,

$$Z_{\Lambda,\varepsilon}(\beta) = \operatorname{Tr} e^{-\beta H_{\Lambda,\varepsilon}} = \sum_\sigma \langle\sigma\,| e^{-\beta H_{\Lambda,\varepsilon}} |\sigma\rangle \geqslant \sum_\sigma \exp(-\beta\langle\sigma|\, H_{\Lambda,\varepsilon}\,|\sigma\rangle) \tag{2.22}$$

where we take as an orthonormal basis the eigenvectors for the Ising part of $H_{\Lambda,\varepsilon}$.

Then since $\langle\sigma|\, K_\Lambda\,|\sigma\rangle = 0$ for those vectors,

$$\sum_\sigma \exp(-\beta\langle\sigma|\, H_{\Lambda,\varepsilon}\,|\sigma\rangle) = \sum_\sigma \exp(-\beta\langle\sigma|\, H_{\text{Ising}}\,|\sigma\rangle) = Z_{\text{ising}} \geqslant e^{\beta\,|\Lambda|} \tag{2.23}$$

where the last inequality follows from taking σ as the state with alternating spins. It follows that

$$\frac{\text{Te } 1}{Z_{\Lambda,\varepsilon}(\beta)} \leqslant \frac{2^{|\Lambda|}}{e^{\beta\,|\Lambda|}} \tag{2.24}$$

We combine this with (2.21) to obtain

$$R_+(\beta, \Lambda) \leqslant \exp\{-\beta[e_0(\varepsilon) + \Delta\,|\Lambda| + |\Lambda|]\}\, 2^{|\Lambda|} \tag{2.25}$$

We need a lower bound of $e_0(\varepsilon)$. By first-order perturbation theory for a two-site model, there is a κ such that

$$-\left\{\frac{\varepsilon}{2^\nu}[\sigma^x(i) + \sigma^x(j)] + \sigma^z(i)\,\sigma^z(j)\right\} \geqslant -1 - \kappa\varepsilon^2 \tag{2.26}$$

Summing over all bonds in Λ, we get

$$e_0(\varepsilon) \geqslant -|\Lambda|\,(1 + \kappa\varepsilon^2) \tag{2.27}$$

Thus we have the following lemma.

Lemma 2.3. There as a κ for which

$$R_+(\beta, \Lambda) \leqslant 2^{|\Lambda|} \exp[-\beta\,|\Lambda|\,(\Delta - \kappa\varepsilon^2)] \tag{2.28}$$

To estimate $R_{-}(\beta, \Lambda)$ we use the following theorem on exponential localization of eigenvectors.

Theorem 2.4 (Frohlich and Lieb[5]). Let A and B be self-adjoint operators on a Hilbert space H such that

(i) $A \geqslant 0$

(ii) $\pm B \leqslant \delta A$

for some δ, $0 \leqslant \delta < 1$. Suppose

$$(A+B)\psi = \lambda\psi, \qquad \|\psi\| = 1$$

Choose $\rho > \lambda$ such that

$$\theta = \delta\rho(\rho - \lambda)^{-1} \leqslant 1$$

Let M_ρ denote the span of eigenvectors of A with eigenvalue $\geqslant \rho$, and let $\phi \in M_\rho$ be a normalized vector such that

(iii) $[B(A-\lambda)^{-1}]^j \phi \in M_\rho$

for $j = 0, 1, \ldots, d-1$ with $d \geqslant 1$. Then

$$|\langle \phi, \psi \rangle| \leqslant \theta^d$$

Moreover,

$$\langle \psi P \psi \rangle = |\langle \psi, \phi \rangle \langle \phi, \psi \rangle| \leqslant \theta^{2d} \tag{2.29}$$

Remark. The theorem says that ϕ and ψ are approximately orthogonal. The intuition is that if

$$P_\Lambda = \psi_\Lambda = \sum a_n \phi_{\lambda_n} \tag{2.30}$$

where ϕ_{λ_n} are eigenvectors of A with eigenvalue $\lambda_n < |\Lambda|$, then

$$a_n = \langle \psi_\Lambda, \phi_{\lambda_n} \rangle \sim e^{-\alpha\,||\Lambda| - \lambda_n|} \tag{2.31}$$

We now apply this theorem to our model. Let

$$A_\alpha = -\Big(\sum_{\substack{i,j \in \Lambda \\ |i-j|=1}} \sigma^z(i)\,\sigma^z(j) - 1 - \kappa\alpha^2 \Big) \tag{2.32}$$

with $\kappa > 0$, and

$$B_\varepsilon = \varepsilon K_\Lambda = -\varepsilon \sum_{i \in \Lambda} \sigma^x(i) \tag{2.33}$$

By the argument used in Eq. (2.26),

$$\pm B_\varepsilon \leqslant \delta A_{\varepsilon/\delta}$$

Let

$$\rho = |\Lambda| \, [2\kappa(\varepsilon/\delta)^2 + 2\Delta] \tag{2.34}$$

Eigenvalues of $H_{\Lambda,\varepsilon}$ in the interval $[e_0(\varepsilon), e_0(\varepsilon) + \Delta\,|\Lambda|]$ correspond to eigenvalues of $B_\varepsilon + A_{\varepsilon/\delta}$ in the interval

$$\begin{aligned}[e_0(\varepsilon) + |\Lambda| \, (1 + \kappa(\varepsilon/\delta)^2), e_0(\varepsilon) + |\Lambda| \, (1 + \kappa(\varepsilon/\delta)^2 + \Delta)] \\ \subset [0, |\Lambda| \, (\Delta + 2\kappa(\varepsilon/\delta)^2)]\end{aligned} \tag{2.35}$$

for $0 < \delta < 1$. For an eigenvalue λ of $B_\varepsilon + A_{\varepsilon/\delta}$ in this latter interval, $\rho - \lambda \geqslant \Delta\,|\Lambda|$ so that θ of the theorem may be taken as

$$\theta = 2\delta \left[\frac{\kappa(\varepsilon/\delta)^2 + \Delta}{\Delta} \right] \tag{2.36}$$

Since $A_{\varepsilon/\delta} - \lambda$ leaves M_ρ invariant, where M_ρ is the space spanned by eigenvectors of $A_{\varepsilon/\delta}$ with eigenvalues $\lambda \geqslant \rho$, part (iii) of the theorem amounts to computing the minimum number of steps required to transform the universal state P_Λ of energy $|\Lambda|$ to a state of energy $<\rho$. At worst, a flip can decrease the energy by $2 \cdot 2^\nu$ quanta, so that

$$d \geqslant \frac{|\Lambda|}{2^{\nu+1}} \, [1 - 2\kappa(\varepsilon/\delta)^2 - 2\Delta] \tag{2.37}$$

Using the theorem, we have the following estimate.

Lemma 2.5. If $1 - 2\kappa(\varepsilon/\delta)^2 - 2\Delta \geqslant 0$, then

$$\langle \phi P_\Lambda \phi \rangle \leqslant \left\{ 2\delta \left[\frac{\kappa(\varepsilon/\delta)^2 + \Delta}{\Delta} \right] \right\}^{|\Lambda| \, [1 - 2\kappa(\varepsilon/\delta)^2 - 2\Delta]/2^\nu} \tag{2.38}$$

where ϕ is any normalized eigenvector of $H_{\Lambda,\varepsilon}$ with eigenvalue $\lambda \in [e_0(\varepsilon), e_0(\varepsilon) + |\Lambda| \, \Delta]$.

An immediate consequence is the following:

Lemma 2.6. For $1 - 2\kappa(\varepsilon/\delta)^2 - 2\Delta \geqslant 0$

$$R_-(\beta, \Lambda) \leqslant \left\{ 2\delta \left[\frac{\kappa(\varepsilon/\delta)^2 + \Delta}{\Delta} \right] \right\}^{|\Lambda| \, [1 - 2\kappa(\varepsilon/\delta)^2 - 2\Delta]/2} \tag{2.39}$$

Proof. We have

$$R_{-}(\beta, \Delta) = \frac{1}{(\sum e^{-\beta\lambda_n})} \left(\sum_{\substack{n \\ \lambda_n \leqslant e_0 + \Delta|\Lambda|}} e^{-\beta\lambda_n} \langle \phi_n P_\Lambda \phi_n \rangle_\Lambda \right) \tag{2.40}$$

where ϕ_n are normalized eigenvectors of $H_{\Lambda\varepsilon}$ with eigenvalue λ_n,

$$\leqslant \sup_{\substack{n \\ \lambda_n \leqslant e_0 + \Delta|\Lambda|}} \langle \phi_n P_\Lambda \phi_n \rangle_\Lambda \leqslant \left\{ 2\delta \left[\frac{\kappa(\varepsilon/\delta)^2 + \Delta}{\Delta} \right] \right\}^{|\Lambda|[1 - 2\kappa(\varepsilon/\delta)^2 - 2\Delta]/2^\nu} \tag{2.41}$$

by the previous lemma. ▮

We now complete the proof of the key estimate of Theorem 2.2. By inequalities (2.28) and (2.41) we have

$$\begin{aligned} \langle P_\Lambda \rangle_\Lambda &= R_{+}(\beta, \Delta) + R_{-}(\beta, \Delta) \\ &\leqslant 2^{|\Lambda|} \exp[-\beta |\Delta| (\Delta - \kappa\varepsilon^2)] + \left\{ 2\delta \left[\frac{\kappa(\varepsilon/\delta)^2 + \Delta}{\Delta} \right] \right\}^{|\Lambda|[1 - 2\kappa(\varepsilon/\delta)^2 - 2\Delta]/2^\nu} \end{aligned} \tag{2.42}$$

At this point we choose

$$\Delta = 2\kappa(\varepsilon/\delta)^2 = 1/6 \tag{2.43}$$

so that (using $\delta < 1$)

$$\begin{aligned} \langle P_\Lambda \rangle_\Lambda &\leqslant 2^{|\Lambda|} \exp\left(-\beta \frac{|\Delta|\Delta}{2} \right) + (3\delta)^{2^{\nu(1-3\Delta)}} \\ &\leqslant 2^{|\Lambda|} \exp\left(-\frac{\beta|\Delta|}{12} \right) + (3\delta)^{|\Lambda|/2^{\nu+1}} \end{aligned} \tag{2.44}$$

Thus for any $\mu > 0$, there is a β_c and a δ_c such that

$$\langle P_\Lambda \rangle_\Lambda^{1/|\Lambda|} < e^{-\mu} \tag{2.45}$$

for $\beta \geqslant \beta_c$ and $\delta < \delta_c$. By Eq. (2.43) $\delta < \delta_c$ is equivalent to

$$\varepsilon < \varepsilon_c = \delta_c (12\kappa)^{-1/2} \tag{2.46}$$

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